
Fractal analysis of tree paintings by Piet Mondrian (1872–1944)

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Abstract: We examine two paintings by Piet Mondrian, and suggest that his depiction of tree foliage exhibits fractal patterns of a specific dimension. Our analysis implies that fractality may possess an aesthetic value that affected Mondrian, perhaps in a similar way as it inspired Jackson Pollock, another famous painter who incorporated fractality in several of his paintings. In recent years there has been a stimulating debate among scientists arguing for and against the thesis that Jackson Pollock's drip paintings when analysed at small scales can be described by the mathematics of fractal geometry. Our suggestion that fractal patterns exist in the paintings of a second famous artist—in this case Piet Mondrian—further supports the hypothesis that the beauty which exists in fractality may affect consciously or subconsciously such great painters as Jackson Pollock and Piet Mondrian.

Keywords: fractals; fractal dimension; tree paintings; Piet Mondrian; scaling laws.

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1 Introduction

In the present paper we argue that the Dutch artist Piet Mondrian, at least with regard to two of his well-known tree paintings, has created fractal patterns that can be characterised by a specific non-integer dimension. In order to compute this dimension we will first examine two paintings of Mondrian under a visual sequence of decreasing scales and then extend our analysis digitally to even smaller scales in order to obtain reliable estimates of the relevant fractal dimensions.

Our study clearly suggests that the two works examined here do possess fractal features. Detailed calculations appear to reveal the feeling of Mondrian that the true nature of certain objects can be understood only by examining them at smaller and smaller scale. This provides a general framework for the aesthetic value of fractality.

Our paper is organised as follows: in Section 2, we discuss Jackson Pollock’s drip paintings and summarise the scientific arguments that support their fractality and lead to the estimation of their fractal dimension. We also describe various counterarguments expressed by a number of scientists regarding the reliability of the dimension calculations, and their use for determining the authenticity of a given painting. In Section 3, we discuss certain philosophical as well as artistic beliefs of Piet Mondrian. We then select two of his most famous paintings (1910 and 1916) to demonstrate that he has also painted pictures that appear to possess fractal characteristics.

In Section 4, we review some basic facts of fractal geometry and describe the standard approach for calculating fractal dimensions, which employ the box counting method also used earlier for Pollock’s paintings. In Section 5, we present the results of a preliminary computation, based on a two-step ‘differential’ approach of visually guided analysis, which suggests that both tree paintings we have analysed indeed possess distinct fractal features. Then, in Section 6 we revisit our calculations, applying a more sophisticated box-counting method which utilises digitally guided image analysis to estimate the dimension of the tree foliage. This section outlines the mathematical

details of our approach and presents specific results as well as estimates of confidence intervals. The main conclusion of our study is that Mondrian's fractals also possess a dimension close to 1.7 which is similar to the dimension of some of Pollock's later drip paintings. Finally, in Section 7 we discuss further our findings emphasising that intriguing questions concerning the properties of fractal patterns are still open. In particular, the existence of subtle connections between art and science remain to be elucidated.

2 Fractal analysis of paintings by Jackson Pollock (1912–1956)

In a paper entitled 'Fractal analysis of Pollock's drip paintings' published in *Nature* (Taylor et al., 1999) analysed a number of paintings, which were claimed to have been created by the famous artist Jackson Pollock in the 1940s, employing a highly elaborate and systematic procedure that involved dripping paint on canvas. Since these paintings were of doubtful originality, the above authors decided to use mathematics to study their intricate patterns aiming to determine the paintings' authenticity. The results reported in Taylor et al. (1999):

- 1 apparently provided strong evidence that many of Pollock's paintings are characterised by fractal geometry (see Section 4 below)
- 2 purportedly offered a reliable method for authenticating whether a particular work has been painted by Jackson Pollock; this method involves estimating the fractal dimension and comparing it with the dimension of paintings known to be created by Pollock (note that the fractal dimension of Pollock's paintings changed over time during the painter's career).

The work of Richard Taylor and his colleagues was questioned in a paper published in *Nature* by Jones-Smith and Mathur (2006). These authors criticised the results of Taylor's analysis by pointing out that the examined paintings exhibited fractal characteristics over a range too small to be reliably considered as fractal. In particular, they argued that similar fractal characteristics could also be found in drawings generated by a Gaussian random process. In the same issue of *Nature*, Taylor et al. (2006) responded to the above criticism stating that their use of the term fractal is consistent with what is generally accepted by the scientific community when physical fractals are studied under limited magnification norms.

In what follows, we summarise some of the main points of the above debate, and then present the results of our study on the fractal patterns observed in some paintings by Piet Mondrian (1872–1944).

In 2009, in a paper published in PRE, Jones-Smith et al. (2009) provided further support for their claim that fractal analysis as used by Taylor and his group cannot be applied to characterise unambiguously works of art. However, it is interesting that in the process of analysing such paintings, these researchers actually discovered some new techniques of fractal mathematics and developed a new process for separating the coloured layers of paint in art works.

There have also been a number of papers defending Taylor's approach. Notably, Coddington et al. (2008), in a paper entitled 'Multifractal analysis and authentication of Jackson Pollock Paintings', argued that fractal geometry can indeed be used to provide a

quantitative analysis for Jackson Pollock's drip paintings. In particular, they introduced the calculation of a quantity called the 'entropy dimension', and discussed the possibility of using this quantity to study the signature of Pollock's works.

More support for Taylor's fractal analysis was provided by Herczynski et al. (2011) who used the physics of fluids to understand how Pollock employed gravity and paint of varying viscosities to make on the canvas coils, splashes and spots. Among other things, these authors showed that the only way Pollock could have generated such tiny loopings and meandering oscillations, was to hold his brush high up above the canvas and let out a flow of paint that narrowed and accelerated as it fell. This suggests that Pollock employed clever physical techniques to aid his art. Remarkably, Pollock created these paintings in the 1940s, long before physicists in the 1950s and 60s worked out the relevant fluid dynamics, and mathematicians in the 1970s and 1980s analysed fractal geometry.

It is interesting to note that Taylor also investigated the connection between neurobiology and fractals. Using MRI, EEG and skin conductance data he was able to establish that looking at fractals reduces human stress levels. Thus, the aesthetic appeal of fractals can be measured physiologically. In a recent paper, Taylor et al. (2011) describe in detail their research over many years showing that the impact of Jackson Pollock's paintings to viewers depends on the paintings' fractal dimension.

The brain employs different types of abstraction. In visual perception, the most elemental abstraction is the deconstruction of an image in terms of just lines and colours. This is best illustrated in the late paintings of Piet Mondrian. Pollock's paintings illustrate a more complex form of abstraction that is based on top-down Gestalt processes. Thus, in contrast to Mondrian's simplicity, Pollock's 'Abstract expressionism' displays complexity in the form of a tangled web of intricate paint splatters. Nevertheless, as we shall show in this paper, Mondrian earlier had also painted complex natural objects, like tree foliages, which may be characterised by fractals. It is well known that the brain deconstructs and reconstructs; perhaps Mondrian, later in his life, deconstructed his earlier complex patterns into simple geometric shapes.

Art theorists categorise the evolution of Pollock's pouring technique into three phases (Varnedoe and Karmel, 1998). In the preliminary phase of 1943–1945, his paintings were characterised by low values of the fractal dimension D , namely $D = 1.3 - 1.5$. During his transitional phase from 1945 to 1947, when he was experimenting with the pouring technique, the D values rose sharply. In his mature period of 1948–1952 the dimension of his paintings reached the value of $D = 1.7$. This observation raises an interesting question: Did these higher D fractal patterns hold a special aesthetic quality for Pollock and, if so, do viewers of his work share the same preference?

Preference experiments involving 119 participants from the general public suggested that the most preferred D value is about 1.3 (Hagerhall et al., 2004), indicating that the preference for mid-range D values of simple fractal shapes (Aks and Sprott, 1996; Spehar et al., 2003) extends to the characteristics of a more intricate fractal scenery. To summarise, perception studies of fractals generated by nature, mathematics, and art, indicate that images in the range $D = 1.3 - 1.5$ have the highest aesthetic appeal.

3 Fractal trees in paintings by Piet Mondrian (1872–1944)

Based on the above discussion, one may wonder whether there are other famous painters, besides Pollock, who although lived before fractal geometry became popular, were somehow inspired to introduce fractal features in some of their artworks. The claim of the present paper is that this was indeed the case with the Dutch artist Piet Mondrian, at least with regard to two of his well-known tree paintings.

Piet Mondrian moved to Paris in 1911, changing his name from Mondriaan, as a sign that he wanted to leave the Netherlands behind him. As is well known, in Paris he was strongly influenced by the cubist style of Picasso and Braque. However, it appears that Cubism influenced only a brief period in his life. Mondrian frequently attempted to reconcile his work with spiritual studies and philosophical explorations that went beyond representational painting. In a letter to H.P. Bremmer in 1914, Mondrian expressed his theory about art as follows:

“I construct lines and colour combinations on a flat surface, in order to express general beauty with the utmost awareness. Nature (or, that which I see) inspires me, puts me, as with any painter, in an emotional state so that an urge comes about to make something, but I want to come as close as possible to the truth and abstract everything from that, until I reach the foundation (still just an external foundation) of things...”

Trees are featured very often in Mondrian’s work in many shapes and forms, illustrating his different approaches to the representation of nature. Besides his attempts at abstraction and reduction of a tree to its most fundamental elements, one also finds wonderful examples of realism that explore the complexity of the tree itself. The painter seems to marvel at the emergence of a multitude of shapes in the form of smaller and smaller branches that ultimately reveal, via a rich variety of scales, the tree’s identity.

We have chosen in this study to focus on two examples of Mondrian’s tree painting, in order to investigate their content in the light of novel mathematical developments that appear to reveal certain hitherto unsuspected connections between geometry and art. The two paintings we are going to consider are the following:

- the Red Tree, 1910, oil on canvas, dimensions: 70 × 99 cm, exhibited at Haags Gemeentemuseum, The Hague, Netherlands
- farm near Duivendrecht, c. 1916, oil on canvas, 86.3 × 107.9 cm, exhibited at the Art Institute of Chicago.

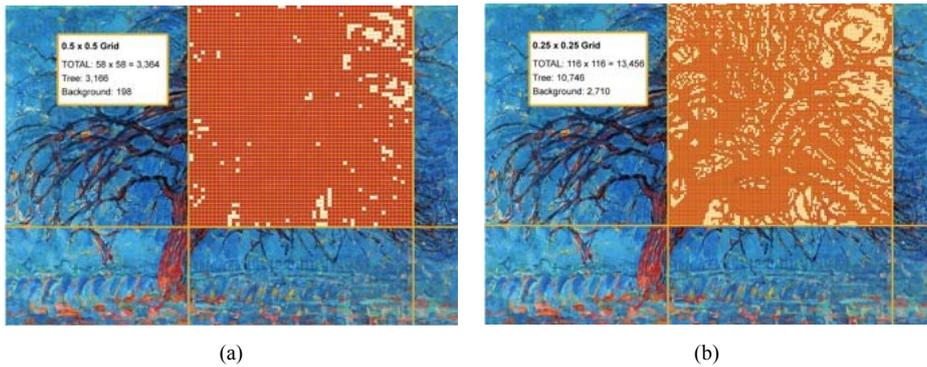
In what follows, we shall call them the Red Tree and the Farmhouse Tree, respectively (see Figures 1 and 2).

What strikes us immediately about these works is the complexity of branching in the Red Tree and in the tree above the farmhouse. It seems that the artist was fascinated by the way bigger branches divide into smaller ones, in an apparently never ending sequence of decreasing scales. It is doubtful that his aim was to illustrate the inventiveness of nature to provide nourishment in an efficient way down to the smallest parts of a tree. Perhaps, what may have attracted him is nature’s way of filling space apparently ‘densely’, using structures that have a much smaller capacity than what the original object implies.

Figure 1 (a) The Red Tree (Mondrian, 1910) and (b) the farm tree (Farm near Duivendrecht, Mondrian, 1916) (see online version for colours)



Figure 2 The Red Tree with a covering of its branches by squares with (a) scale $0.5u$ and (b) scale $0.25u$ respectively (see online version for colours)



Fractal geometry is the part of mathematics that studies precisely such patterns, in which the employment of smaller and smaller scales gives rise to the remarkable fact that these structures can occupy space greater than what the overall figure entails. Are such structures beautiful? Well, many artists thought so after fractal geometry became popular in the beginning of the 1980s. A rich variety of exhibitions in the last 30 years all over the world have attempted to highlight such a connection.

However, discovering relations between fractality and beauty in earlier artists, would indeed be surprising. And yet, as discussed in detail in Section 2, several paintings by the famous artist Jackson Pollock (1912–1956) can be shown to have a fractal structure that can actually be measured quantitatively. Hence, it does not appear futile to look for similar relations in other artists, contemporaries of Pollock like Piet Mondrian, aiming to establish that a purely geometrical property, such as fractality, is not only useful for performing Nature’s functions but may also have a deeper aesthetic value.

4 Mathematical considerations of fractal geometry

The fundamental property of a fractal object (or fractal for short), is that it consists of parts of smaller and smaller magnitude, whose total contribution can be estimated by covering the object by an increasing number of ‘boxes’ of smaller and smaller size (Barnsley, 1993). If the object is embedded in one dimension ($d = 1$) then these ‘boxes’ are linear segments of length l , if it lies in two dimensions ($d = 2$) the ‘boxes’ are squares of side l , in three dimensions ($d = 3$) they are cubes of side l , etc.

Obviously, the more ‘boxes’ we use the smaller their size length and the greater their number $\mathcal{N}(l)$. Their contribution to the total ‘measure’ of the object M (= ‘length’ when $d = 1$, ‘area’ if $d = 2$, ‘volume’ if $d = 3$, etc.) is given by the formula

$$M = \lim_{\substack{l \rightarrow 0 \\ \mathcal{N}(l) \rightarrow \infty}} \mathcal{N}(l)l^D. \quad (1)$$

In equation (1) one takes the limit of l going to zero and $\mathcal{N}(l)$ going to infinity. It is important to note that the power D to which the size length is raised corresponds to the dimension of the object. This is the most crucial parameter characterising fractality, and in what follows we will focus on the estimation of D . Its value is expected to satisfy $d - 1 < D \leq d$, i.e., smaller or equal to the dimension of the space where the fractal has been embedded. If $D = d$ the object occupies fully the space in which it lies, and one speaks about an ordinary object of integer dimension, consistent with what is known from everyday experience.

However, for a fractal object we expect that $D < d$, meaning that the structure under study may seem to extend in d dimensions, but in fact it actually occupies a space of dimension smaller than d . This appears strange, as it clearly implies the existence of spaces of *non – integer* dimensions. However, in Mathematics one often has no other choice but to accept the implications of one’s findings, however strange they may seem, provided they lead to logical conclusions. Indeed, there are objects in our world, for which if one sets $D = d$ in equation (1) one finds $M = 0$, while $D = d - 1$ gives $M = \text{infinity}$. Clearly both of these results are absurd and thus one is forced to choose that value of D – unique according to the theory of fractal geometry (Barnsley, 1993) – that gives a precise finite value to M in equation (1).

In practice, of course, one cannot take l to be arbitrarily small, neither can $\mathcal{N}(l)$ become arbitrarily large. Thus, one selects a sequence of decreasing values of l_n and respectively increasing values $\mathcal{N}(l_n)$, $n = 1, 2, 3, \dots$ and calculates the dimension D of the object from the formulae:

$$M \approx \mathcal{N}(l_1)l_1^D \approx \mathcal{N}(l_2)l_2^D \approx \mathcal{N}(l_3)l_3^D \approx \dots \quad (2)$$

This is certainly an approximate estimate, as indicated by the curly equality signs in equation (2). The above estimate becomes better as the size lengths become smaller and the number of ‘boxes’ becomes larger. In order to make the first steps towards calculating D we will proceed to solve the above approximate equalities as true equalities, using three coverings by boxes of length size

$$l_1 = 0.75u, \quad l_2 = 0.5u, \quad l_3 = 0.25u \quad (3)$$

where u represents a common length unit in all our measurements.

Our strategy, therefore, is as follows: We first select a big rectangle containing part of the branches of each of the two Mondrian's trees, and divide it into a grid of much smaller squares whose side lengths are successively l_1 , l_2 , l_3 according to equation (3) (see Figures 2 and 3). For each of these situations, we will denote by $\mathcal{N}(l_n)$, $n = 1, 2, 3$, the number of boxes that contain part of the tree inside them by counting first all the empty boxes and then subtracting them from the total number of squares contained in the big rectangle. Then we solve separately for each painting the following equations:

$$\mathcal{N}_1(0.75u)^D = \mathcal{N}_2(0.5u)^D, \quad \mathcal{N}_2(0.75u)^D = \mathcal{N}_3(0.25u)^D, \quad (4)$$

eliminating from them the unit length u , which is irrelevant. Ideally, one would expect that the values of D satisfying the above two equations would coincide, since they both lead to the same measure of the tree M . This will of course not happen, since our approximations are quite crude. It will be very interesting, however, if the following occur:

- a these values are close and lie well below 2
- b as we move from the larger scales to the smaller ones, the dimension changes in a way consistent with fractality (to be explained below).

Figure 3 The Farmhouse Tree with a covering of its branches by squares with scale (a) $0.5u$ and (b) $0.25u$ respectively (see online version for colours)



Before displaying our calculations, let us describe what we expect to find: Since, in both the Red Tree and the Farmhouse Tree, the pictures of the trees look quite *sparse*, with wide open spaces between the branches, one would suspect that the trees cover *only a small part* of the 2-dimensional plane (here, clearly $d = 2$). Thus, if the object has a significant fractal structure, the dimension should have a value $D < 2$, even at small scales. Of course, at extremely small scales (since the branches are not simple lines), the tree will turn out to be 2-dimensional. However, if the object is far from being a fractal the dimension $D = 2$ should be evident already at the small scales. As we shall find out this does not happen. The values of the dimension we find *are clearly less than 2* and demonstrate that, at least down to the scales we have studied, the two trees can be characterised as fractals with a $D \approx 1.75$ in both cases.

5 A visually guided calculation of the fractal dimension of Mondrian's trees

Let us take the Red Tree of Figure 1 and isolate a rectangular part of its body, which we divide in a grid of squares of size $l_1 = 0.75u$ in some units. Subtracting the empty boxes from the total number, we find that the tree is covered by $\mathcal{N}_1 = 1494$ boxes (not shown here). We then move to a division of the rectangular area by boxes of size $l_2 = 0.5u$, in which case we find that the tree is covered by $\mathcal{N}_2 = 3,166$ boxes (as shown in Figure 3). Thus, performing the first calculation of the dimension using the first of equation (4), we find

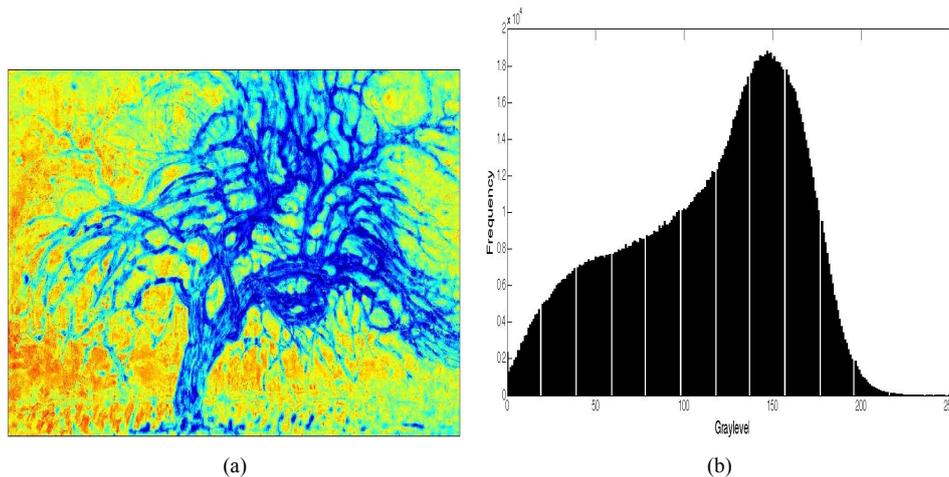
$$\frac{\mathcal{N}_2}{\mathcal{N}_1} = \left(\frac{0.75}{0.5}\right)^D \Rightarrow \frac{3166}{1,494} = \left(\frac{0.75}{0.5}\right)^D \Rightarrow D = 1.85. \quad (5)$$

Thus, the first estimate for the dimension of the Red Tree is 1.85, which is well below 2, but not far from it. Note that at large scales the tree looks rather dense. It is, therefore, crucial to proceed to the next calculation at which the sparsity of the branches should become more apparent and the dimension of the object, if it has fractal features, should become smaller. Indeed, performing the calculation we find (see Figures 3 and 4)

$$\frac{\mathcal{N}_3}{\mathcal{N}_2} = \left(\frac{0.5}{0.25}\right)^D \Rightarrow \frac{10746}{3166} = \left(\frac{0.5}{0.25}\right)^D \Rightarrow D = 1.75. \quad (6)$$

This is encouraging, and provides clear evidence of fractality of the Red Tree, at least at the scales studied here.

Figure 4 (a) The blue component of the Red Tree painting and (b) its histogram (see online version for colours)



Let us now repeat the calculation for the Farmhouse Tree. As is clear in Figure 2, the tree in the middle of the painting looks quite sparse at large scales even at its upper part, where we begin to search for fractality. Therefore, it would be natural to expect

that the first estimate of its dimension would be lower than what is visible at smaller scales. Indeed, when we perform the first calculation we obtain

$$\frac{\mathcal{N}_2}{\mathcal{N}_1} = \left(\frac{0.75}{0.5}\right)^D \Rightarrow \frac{2,848}{1,478} = \left(\frac{0.75}{0.5}\right)^D \Rightarrow D = 1.62. \quad (7)$$

Proceeding to the smaller scales, as shown in Figures 5 and 6, we find

$$\frac{\mathcal{N}_3}{\mathcal{N}_2} = \left(\frac{0.5}{0.25}\right)^D \Rightarrow \frac{9,604}{2,848} = \left(\frac{0.5}{0.25}\right)^D \Rightarrow D = 1.75, \quad (8)$$

a result quite similar to what was found earlier for the Red Tree.

6 Estimation of the fractal dimension using box counting and linear regression

Self-similarity, *compass-dimension* and *box-counting*, are all special features of Mandelbrot's fractal dimension which was originally motivated by Hausdorff's fundamental work (Peitgen and Saupe, 2004). In order to compute the fractal dimension of a self-similar structure one often adopts a sequential limiting process. Specifically, one puts the structure into a grid of mesh size S and counts the number of boxes which contain some of the structure. If this number is denoted by $\mathcal{N}(S)$, the following limit constitutes the desired quantity we are looking for:

$$D = \lim_{S \rightarrow 0} \frac{\log(\mathcal{N}(S))}{\log(S^{-1})}. \quad (9)$$

In practice, of course, we cannot compute the above mentioned limit since the scale S in a given binary image cannot take arbitrarily small values. More specifically, the smallest allowed value we can use for the scale parameter is strongly dependent on the sampling frequency used during the capturing process of the image. Moreover, the equality in equation (9) must be replaced by the symbol of proportionality. Thus, we consider a decreasing sequence of scales S_k , $k = K - 1, \dots, 1, 0$, and form the following sequence:

$$D_k = \frac{\log(\mathcal{N}(S_k)) - \log(M)}{\log(S_k^{-1})}, \quad k = K - 1, \dots, 1, 0, \quad (10)$$

where M denotes the constant of proportionality in the power law which is expressed by equation (9). Note that each member of this sequence D_k , $k = K - 1, \dots, 1, 0$, can be considered as an estimation of the desired fractal dimension. More specifically, under error free conditions and assuming a monotone convergence of the above sequence, it is clear that the last member of the sequence will be the best achievable one. However, each term of the sequence is not directly computable owing to its dependence on the unknown proportionality constant M . Thus, in order to overcome this difficulty we rewrite equation (10) in the form

$$\log(\mathcal{N}(S_k)) = \log(S_k^{-1})D_k + \log(M), \quad k = K, \dots, 1, 0. \quad (11)$$

We could estimate the desired dimension by applying the difference operator $\mathcal{D}\{\cdot\}$ on both sides of (11) and keep the last member of the resulting sequence, i.e.,

$$D \approx \frac{\log\left(\frac{\mathcal{N}(S_1)}{\mathcal{N}(S_0)}\right)}{\log\left(\frac{S_0}{S_1}\right)} = \hat{D}. \quad (12)$$

This was essentially done in Section 5; here we prefer to employ a more accurate approach and solve the following linear regression problem:

$$\min_{\alpha, \beta} \sum_{k=0}^{K-1} \|\log(\mathcal{N}(S_k)) - \alpha \log(S_k^{-1}) - \beta\|_2^2, \quad (13)$$

where $\|\mathbf{x}\|_2$ denotes the l_2 norm of the vector \mathbf{x} . It is shown in (Douglas, E.A.P. et al., 2012) that the optimal estimators resulting from the solution of the above defined optimisation problem are given by the following equations:

$$\alpha^* = \hat{D} = \frac{\sigma_{XY} - \mu_X \mu_Y}{\sigma_X - \mu_X^2}, \quad \beta^* = \log(\hat{M}) = \frac{\mu_Y \sigma_X - \mu_X \sigma_{XY}}{\sigma_X - \mu_X^2}. \quad (14)$$

In these equations, μ_X and μ_Y are defined by

$$\mu_X = \frac{\log\left(\prod_{k=0}^{K-1} S_k\right)}{K}, \quad \mu_Y = \frac{\log\left(\prod_{k=0}^{K-1} \mathcal{N}(S_k)\right)}{K}, \quad (15)$$

as the logarithms of the geometric means of the scale sequence S_k , $k = 0, 1, \dots, K-1$, and the corresponding sequence of the number of boxes $\mathcal{N}(S_k)$, $k = 0, 1, \dots, K-1$, which are needed to cover the image, and σ_X , σ_Y , σ_{XY} , defined by

$$\begin{aligned} \sigma_X &= \frac{\sum_{k=0}^{K-1} \log^2(S_k)}{K}, & \sigma_Y &= \frac{\sum_{k=0}^{K-1} \log^2(\mathcal{N}(S_k))}{K}, \\ \sigma_{XY} &= \frac{\sum_{k=0}^{K-1} \log(S_k) \log(\mathcal{N}(S_k))}{K}, \end{aligned} \quad (16)$$

are the mean squared values and the mean cross product values of the logarithms of the above mentioned sequences.

The outline of the algorithm we have employed is as follows:

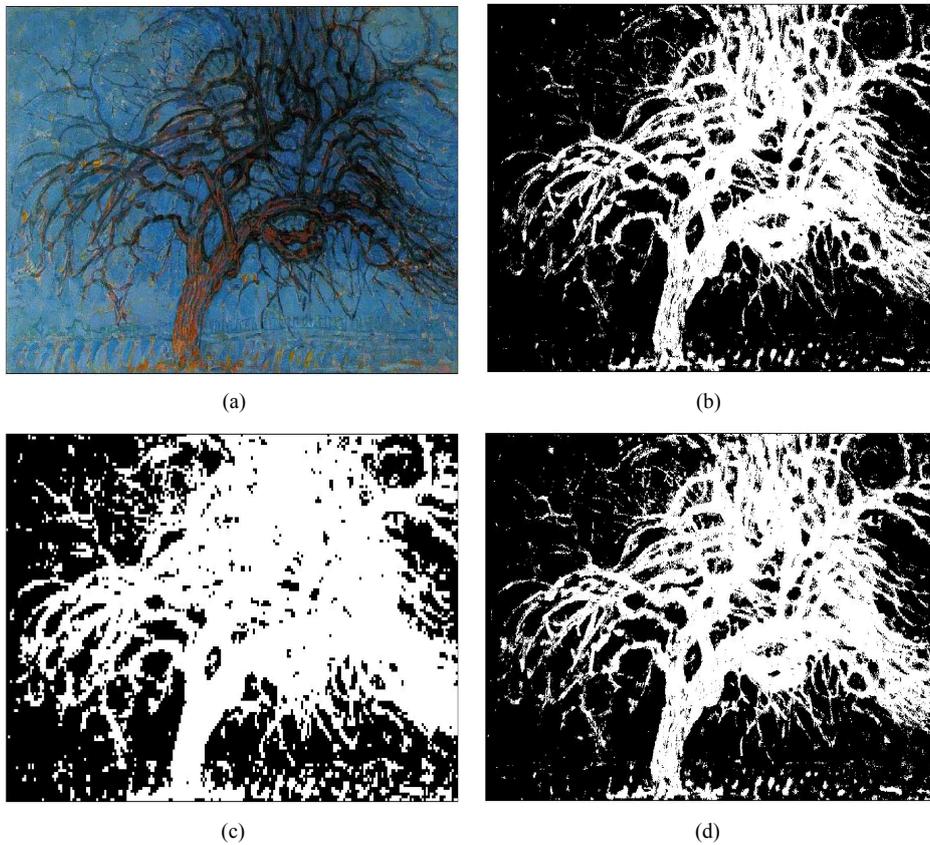
Algorithm 1 Computing the fractal dimension by using the box counting approach and linear regression

Input: the binary image I

- 1 Form the sequence $S_k = 2^k$, $k = 0, 1, \dots, K-1$.
 - 2 For each S_k count the number of boxes $\mathcal{N}(S_k)$ that are needed to cover the image I .
 - 3 Solve the linear regression problem (13); specifically, use equations (14) to (16) to compute the point estimators of fractal dimension \hat{D} and y -intercept \hat{M} .
 - 4 Output: the quantities \hat{D} and \hat{M} as well as related statistics.
-

We have applied the above algorithm to the Mondrian paintings Red Tree [Figure 5(a)] and Farmhouse Tree [Figure 6(a)]. For the binarisation of the colour images, the histograms of their blue components were properly thresholded by using Otsu's (1979) detector. The blue component of the Red Tree image, its histogram, and the resulting binarised image after its thresholding, are shown in Figures 4(a), 4(b) and 5(b) respectively.

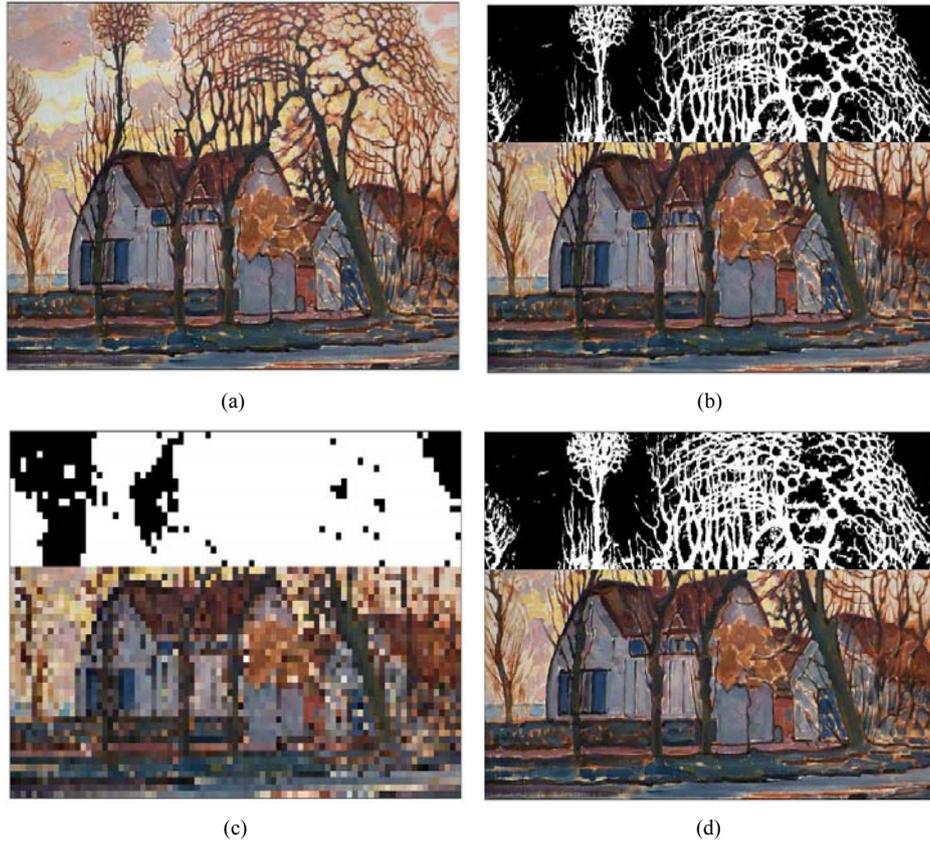
Figure 5 (a) The Red Tree painting, (b) its binarised form (or equivalently its covering for the scale S_0) given as input in the box counting algorithm, and the obtained coverings for the scales (c) $S_3 = 8$ and (d) $S_1 = 2$ respectively (see online version for colours)



The results obtained from the application of the box counting algorithm (Steps 1 and 2 of Algorithm 1) in terms of the number of boxes that are needed to cover each one of the images, for a number of different scales, are contained in Table 1. In addition, the regained coverings of the Red Tree and Farmhouse Tree paintings, for two different scales, are shown in Figures 5(c), 5(d), 6(c) and 6(d) respectively.

The datasets, as well as the optimal fitted regression lines obtained from the solution (Step 3 of Algorithm 1) of the corresponding linear regression problems, are shown in Figure 7. From this figure the quality of the achieved fitting, at least visually, is evident.

Figure 6 (a) Part of the Farmhouse Tree painting, (b) the binarised part of the image given as input in the box counting algorithm, and the obtained coverings for the scales (c) $S_5 = 32$ and (d) $S_1 = 2$ respectively (see online version for colours)



Note: For the sake of better presentation, the binarised part of the image as well as the regained coverings are properly superimposed on the original painting.

The estimated fractal dimension \hat{D} and the estimated y -intercept \hat{M} for each one of the aforementioned paintings, as well as their 99% confidence intervals are contained in the second and the third column respectively of Table 2. By taking into account that a confidence interval provides a measure of precision for the estimated regression coefficient, the statistical significance of the achieved optimal estimations is strongly supported.

In order to test further the significance of regression, as well as quantify the quality of the estimated model, we have used the analysis of variance approach (Douglas et al., 2012). In particular, the coefficient of the determination r^2 , the \mathcal{F} -test statistic with its p -value, and the estimation of the error variance $\hat{\sigma}^2$, at the same level of significance as the one mentioned above, were computed; these values are contained in the same table. From the computed values of the r^2 statistic we can see that our observations are well replicated by the linear model. In particular, the r^2 values

are both equal to 0.999, indicating that 99.9% of the variation in one variable may be explained by the other. This ensures that there exists a strong linear relationship between the ‘observed’ responses (sequence $\log(\mathcal{N}(S_k))$, $k = 0, 1, \dots, K - 1$) and the ‘observed’ data (sequence $\log(S_k^{-1})$, $k = 0, 1, \dots, K - 1$). The validity of this relationship is also verified from the values of \mathcal{F} -test and its p values, as well as from the error variances obtained from the aforementioned regression analysis. Finally, we must emphasise the remarkable similarity of the statistics results computed for the two different paintings.

Figure 7 Plots of $\log(\mathcal{N}(S_k))$ vs. $\log(S_k^{-1})$ according to equation (11), for the data sequences hosted in Table 1 (shown with red and black filled dots) and the computed optimal fitted regression (shown with red and black dotted) lines (see online version for colours)

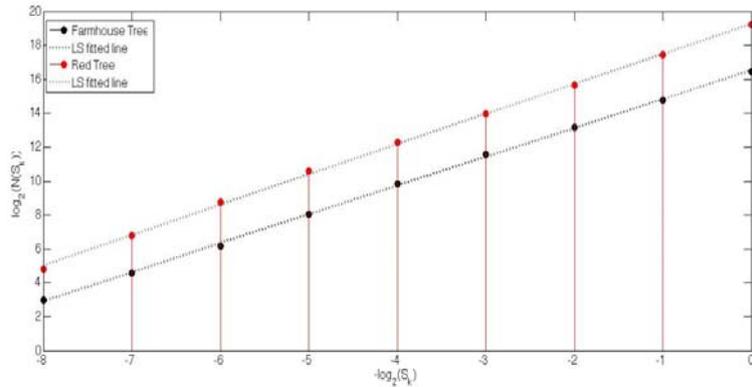


Table 1 Results obtained from the application of the Algorithm 1 to the paintings Red Tree and Farmhouse Tree

Scales: S_k	Number of boxes $\mathcal{N}(S_k)$	
	Red Tree	Farmhouse Tree
256	32	8
128	119	24
64	445	73
32	1,535	267
16	4,928	917
8	15,836	3,018
4	52,265	9,141
2	175,925	28,094
1	622,674	91,121

Table 2 The estimated fractal dimensions and y -intercepts, as well as related statistics obtained from the solution of the linear regression problem (13) for the paintings Red Tree and Farmhouse Tree

Painting	$\hat{D} - \hat{M}$	99% Confidence Interval	r^2	\mathcal{F} -test	p -value	$\hat{\sigma}^2$
Red Tree	1.76–19.25	1.72–1.81 —19.04–19.47	0.999	1,8540	3×10^{-3}	1×10^{-2}
Farmhouse Tree	1.70–16.53	1.66–1.74 —16.33–16.73	0.999	19475	2.5×10^{-3}	9×10^{-3}

7 Conclusions

The above analysis suggests that Piet Mondrian, at some point in his life, found it appealing to paint structures with a degree of fractal complexity that may even be described by an approximate dimension of $D = 1.75$ (Red Tree) or $D = 1.7$ (Farmhouse Tree).

Our observations are valid only for the investigation of two of Mondrian's paintings and only under the limitations of our study and the available resolution of these two paintings. One must keep in mind that a statement concerning the fractality of a certain figure can become definitive only if one is able to examine it in great detail, using a sequence of scales that become progressively smaller. In this direction, we have employed colour information and have extended our digital analysis to a sequence of scales that yield a reliable estimate of a dimension as the slope of a straight line.

The sophisticated mathematical analysis of this study suggests that the two works examined here do possess fractal features. It is evident from our calculations that the painter may have attempted (perhaps unconsciously) to express his feeling that the true nature of some objects can only be revealed when one looks at them more closely. Thus, in order to capture their true identity, one must paint them at smaller and smaller scales. This clearly lies at the heart of what we call fractality.

Does this mean that fractal structures are beautiful as a result of some aesthetic value inherent in their geometry? Certainly, Piet Mondrian, who lived long before fractals became popular, did recognise their beauty; and perhaps, in his efforts to express the truth in nature, he perceived fractality as the perfect vehicle for depicting this truth.

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